Time-Fractional SPDE

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16th International Workshop on Markov Processes and Related Topics

Joint works with Michael Röckner and José Luís da Silva

Outline

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Fractional Derivative: nonlocal

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Motivation:

$$(I^n f)(t) = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} f(s) ds \ (\textbf{Cauchy formula})$$

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- Mechanics (theory of viscoelasticity and viscoplasticity)
- Bio-chemistry (modelling of polymers and proteins)
- Electrical engineering (transmission of ultrasound waves)
- Medicine (modelling of human tissue under mechanical loads)

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♣ Metzler/Klafter, The random walks guide to anomalous diffusion: a fractional dynamics approach. Phys. Rep. 339(2000), 1-77.

Time-fractional derivative (memory effects)

We consider SPDE with time-fractional derivative

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$$X(t) = X(0) - \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}A(s, X(s))ds + \frac{1}{\Gamma(1+\beta-\gamma)}\int_0^t (t-s)^{\beta-\gamma}B(s)dW(s).$$

SPDE with time-fractional derivative

- X. Zhang [JFA '10]: Stochastic Volterra equations
- Cui/Yan [J.Phys.A. '11] Sakthivel et al [NA '13]: neutral/delay ($\gamma = 1$)
- Chen Z./Kim/Kim [SPA '15]: L^2-theory
- Mijena/Nane [SPA '15; PotA '16]: space/time-fractional SHE ($\gamma = \beta$)
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Motivations: Well-posedness of Quasi-linear SPDE

Fractional (stochastic) porous medium equation $(e.g.\Psi(x) = x^m, m > 0)$

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♣ Vergara/Zacher [SIAM JMA '15]: decay estimates

(assuming the existence of solutions)

Outline

Main assumptions $V \subset H \subset V^*$

$$\partial_t^\beta X(t) + A(t, X(t)) = \partial_t^\gamma \int_0^t B(s)dW(s), \ X(0) = X_0, \tag{3}$$

where $A : [0, \infty) \times V \to V^*, B : [0, \infty) \to L_{HS}(U; V)$.

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$(H1)$ (Hemicontinuity) $\quad s \mapsto {}_{V^*}\langle A(t, v_1 + sv_2), v \rangle_V$ is continuous on $\mathbb{R}$.

$(H2)$ (Monotonicity) $\quad {}_{V^*}\langle A(t, v_1) - A(t, v_2), v_1 - v_2 \rangle_V \geq 0$.

$(H3)$ (Coercivity) $\quad {}_{V^*}\langle A(t, v), v \rangle_V \geq \delta \|v\|_V^\alpha - g(t)$.

$(H4)$ (Growth) $\quad \|A(t, v)\|_{V^*} \leq g(t)^{\frac{\alpha-1}{\alpha}} + C\|v\|_V^{\alpha-1}$.

Here $\alpha > 1$, $\delta > 0$ and $g \in L^1([0, \infty); \mathbb{R}_+)$.

Theorem (L./Röckner/Silva, SIAM J. Math. Anal. 2018)

Suppose that A *satisfies* $(H1)$-$(H4)$*,* $B \in L^{\infty}([0,T], L_{HS}(U;V))$ *if* $\gamma < \beta + \frac{1}{2}$ *or* $B \in L^2([0,T], L_{HS}(U;V))$ *if* $\gamma \leq \beta$*, then for every* $X_0 \in L^{\alpha}(\Omega, \mathcal{F}_0; V)$ *(3) has a unique* $\mathcal{F}_t$*-adapted solution and satisfies the following regularity property:*

$$X \in L^{\alpha}([0,T]; V); \ \partial_t^{\beta} X \in L^{\frac{\alpha}{\alpha-1}}([0,T]; V^*), \ \mathbb{P} - a.s.$$

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Applicable examples:

$$A(\cdot,u) = -\Delta(|u|^{r-1}u) \quad (r > 1);$$

$$A(\cdot,u) = -\mathbf{div}(|\nabla u|^{p-2}\nabla u) \quad (p > 1);$$

$$A(\cdot,u) = (-\Delta)^\alpha u \quad (\frac{1}{2} < \alpha \leq 1).$$

Main idea of the proof

$\diamondsuit$ Solvability of $\mathcal{A}u - \Lambda u = f$

where $\mathcal{A}$ is pseudo-monotone/coercive, Λ satisfies admissible condition.

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♠ W. Stannat, *The theory of generalized Dirichlet forms and its applications in analysis and stochastics.* Mem. AMS **142** (1999), 101 pp.

$\Rightarrow$ generalized time-fractional derivatives

Consider SPDE with generalized time-fractional derivatives

$$\partial_t^{*k} X(t) + A(t, X(t)) = \partial_t^{*k_1} \int_0^t B(s) dW(s), \ X(0) = X_0, \qquad (4)$$

where

$$\partial_t^{*k} u := \partial_t (k * u) := \frac{d}{dt} \int_0^t k(t-s) u(s) \, \mathrm{d}s, \quad t \in [0, \infty),$$

for $k \in L^1_{\text{loc}}([0, \infty))$, $k \geq 0$, non-increasing and right-continuous.

Examples of time-fractional derivatives

♠ Liouville/Caputo derivative $k(t) := g_{1-\beta}(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$.

♠ Distributed order derivative $k(t) := \int_0^1 g_\beta(t)\,\mathrm{d}\beta$.

♠ Exponential weight derivative

$$k(t) := g_{1-\beta}(t)e^{-\lambda t} = \frac{t^{-\beta}}{\Gamma(1-\beta)}e^{-\lambda t}.$$

♠ Multi-term derivative $k(t) := g_{1-\beta}(t) + g_{1-\alpha}(t)$.

♠ $\cdots\cdots$

⇒ Weakly monotone case

$$\partial_t^{*k} X(t) + A(t, X(t)) = \partial_t^{*k_1} \int_0^t B(s)dW(s), \ X(0) = X_0, \qquad (5)$$

where $A : [0, \infty) \times V \to V^*, B : [0, \infty) \to L_{HS}(U; V)$.

$(H2)$ (Weak Monotonicity)

$$_{V^*}\langle A(t, v_1) - A(t, v_2), v_1 - v_2 \rangle_V \geq -C\|v_1 - v_2\|_H^2.$$

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♣ Existence and uniqueness: [L./Röckner/Silva, JFA 2021]

Main idea of the proof

Key I: $-\partial_t^{*k}$ is the generator of following C_0-semigroup

$$U_t^k f := \int_0^\infty U_s f \ \mu_t^k(\mathrm{d}s) = f * \mu_t^k, \quad t \geq 0,$$

where $\mu_t^k \leftarrow \psi^k$(Bernstein function) $\leftarrow M^k$(measure) $\leftarrow k$.

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Key II: $-\partial_t^{*k}$ on a properly weighted L^2-space is strongly dissipative

$$\int_0^\infty {}_{V^*}\langle \partial_t^{*k} u(s), u(s)\rangle_V e^{-\gamma s} ds \geq \frac{1}{2}\psi^k(\gamma) \int_0^\infty \|u(s)\|_H^2 e^{-\gamma s} ds.$$

Acknowledgement

Thank you for your attention!

Reference

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